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Freudenberg, James S.; Middleton, Richard H.; Braslavsky, Julio H. "Minimum variance control over a Gaussian communication channel". Published in IEEE Transactions on Automatic Control Vol. 56, Issue 8, p. 1751-1765 (2011)

Available from: <http://dx.doi.org/10.1109/tac.2010.2097831>

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Minimum Variance Control over a Gaussian Communication Channel

J. S. Freudenberg, R. H. Middleton, and J. H. Braslavsky

Abstract—We consider the problem of minimizing the response of a plant output to a stochastic disturbance using a control law that relies on the output of a noisy communication channel. We discuss a lower bound on the performance achievable at a specified terminal time using nonlinear time-varying communication and control strategies, and show that this bound may be achieved using strategies that are linear. We also consider strategies that are defined over an infinite horizon that may achieve better transient response than those that are optimal for the terminal time problem.

I. INTRODUCTION

Recent years have seen much interest in the limitations imposed on a feedback system by the presence of a noisy communication channel in the feedback path, as depicted in Figure 1. One problem is to determine the minimal channel capacity required to stabilize an open loop unstable plant. The solution to this problem is known for noise-free data rate limited channels [24], [25], [29] and additive Gaussian noise channels [6]. A more difficult problem is that of determining the optimal performance, in terms of disturbance attenuation, that is achievable given the presence of a noisy channel with fixed capacity.

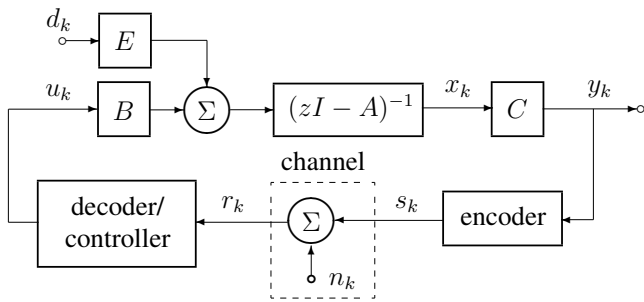


Fig. 1. Feedback control over a noisy communication channel.

The standard minimum variance control problem consists of minimizing the variance of a plant output in response to a stochastic disturbance using a control law that depends on possibly noisy measurements of that output. A solution to this problem, in the case of a noise free measurement, is presented in [2], wherein transfer function methods are used to obtain the result. An alternate approach, that is applicable with noisy

measurements, is to solve the “cheap control” Linear Quadratic Gaussian (LQG) problem, in which a state feedback gain is applied to an estimate of the plant state obtained from a Kalman filter.

In the present paper, we assume that the system output must be communicated to the controller over a Gaussian communication channel. In this scenario, it may be feasible to add precompensation (an encoder) before the channel. For example, we may transmit a filtered version of the system output, or a signal that depends on measurements of the plant states, if these are available. The only restriction is that the channel input must satisfy the power limit of the Gaussian channel. The flexibility available from channel precompensation does not come without a price: the certainty equivalence and separation properties present with LQG optimal control may no longer be present, thus complicating the design of communication and control strategies.

A special case of the minimum variance communication and control problem described above was treated in [11]. In that paper, it was assumed that the channel input is equal to a constant scalar multiple of the plant output, and that the control input is obtained by passing the channel output through a linear time invariant filter. In the present paper, we consider potential improvements using more general communication and control strategies.

Other researchers have studied feedback control performance over a communication channel; a partial review follows. The authors of [22] derive a lower bound on a measure of disturbance attenuation that is stated in terms of channel capacity; however, the proof does not invoke causality at any point, and hence the lower bound may not be tight. In [7], [8] the authors prove a type of separation principle using an encoder with access to feedback from the channel output. The authors of [30] study performance limitations imposed by a vector Gaussian channel, with one channel for each state of the plant. The author of [10] relates the problem of feedback stabilization over a communication channel to that of communication over a channel with feedback. Performance limitations imposed by noise free, data rate limited channels are addressed in [23], [26]. The authors of [3] study the joint optimum design of communication and control strategies for feedback over noisy channels, and show that linear strategies are optimal only for first order linear systems with Gaussian noise and quadratic cost. Another line of research, not pursued in the present paper, concerns limitations imposed by packet-dropping in communication networks; see the recent survey [17].

The remainder of this paper is outlined as follows. In

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Section II we present models of the linear plant, Gaussian channel, encoder, and decoder/controller that we shall study, and pose the problem of minimizing the mean square plant output at a specified terminal time with no penalty on the transient response or the control input. We show that the optimal control at the last time step sets the expected value of the plant output at the terminal time, conditioned on the sequence of channel outputs, equal to zero; the mean square plant output is thus equal to the variance of the optimal estimation error. We then note that the encoder design problem reduces to communicating a “message” that depends on the primitive random variables. In Section III, we derive a lower bound on achievable performance by *temporarily* assuming that the encoder has access to the plant state, the control input, and the channel output, and deriving linear communication and control strategies that use this information to transmit the message defined in Section II over the channel. We use information theoretic arguments in Section IV to show that the resulting error is identical to the minimum possible for any potentially nonlinear communication and control strategies. In Section V we remove the assumption that the encoder has access to the control input and to feedback from the channel output. To do so, we use a control input with a time varying gain that is synchronized to the time varying gain that the encoder uses to form the channel input. In Section VI we make an additional hypothesis that allows us to remove the assumption that the encoder has access to the state of the plant; instead, the input to the encoder is the output of a linear filter whose input is the plant output. The communication and control strategies that minimize the terminal time cost have time varying gains, may yield poor transient response, and cannot be implemented over an infinite horizon. Hence, in Section VII we consider time invariant communication and control strategies that, under appropriate hypotheses, result in the system output becoming stationary. These strategies are suboptimal for the original problem of finite horizon optimal control with a terminal constraint only. However, we show by example that the transient response may be much improved. In developing the infinite horizon strategies, we are required to study the solutions to a class of signal to noise ratio (SNR) constrained Riccati difference equations. Under appropriate hypotheses, the solution to such an equation converges to that of a corresponding SNR constrained algebraic difference equation. Conclusions and directions for further research are presented in Section VIII.

Notation and Terminology

We use upper case letters to denote random variables, lower case letters to denote realizations of these random variables, subscripts to denote elements of a sequence, and superscripts to denote subsequences, e.g., $x^k \triangleq \{x_0, x_1, \dots, x_k\}$. Denote the expected value of the random variable X by $\mathcal{E}\{X\}$. Given two random variables X and Y , denote the conditional expectation of X given that $Y = y$ by $\mathcal{E}_y\{X\} = \mathcal{E}\{X|Y = y\}$, and the associated random variable [28] by $\mathcal{E}_Y\{X\}$. The “smoothing property” of conditional expectations [16, p. 498], [28, p. 123] states that $\mathcal{E}\{X\} = \mathcal{E}\{\mathcal{E}_Y\{X\}\}$. It is well

known (cf. [16, p. 504], [18, p. 97]) that the conditional expectation $\mathcal{E}_Y\{X\}$ minimizes the variance of the mean square estimation error with respect to all other functions $g(Y)$: $\mathcal{E}\{(X - \mathcal{E}_Y\{X\})^2\} \leq \mathcal{E}\{(X - g(Y))^2\}$.

Denote the open unit disk by \mathbb{D} . A linear system $x_{k+1} = Ax_k + Bu_k$, $y_k = Cx_k$ is *stable* if all eigenvalues of A lie within \mathbb{D} , and is *strictly minimum phase* if all zeros of its transfer function $G(z) = C\Phi(z)B$ lie within \mathbb{D} , where $\Phi(z) \triangleq (zI - A)^{-1}$. The *relative degree* of $G(z)$ is equal to its excess of poles over zeros. We say that (A, B) is *reachable* or *stabilizable* and that (A, C) is *observable* or *detectable* according to standard definitions [1, pp. 341–342].

II. PRELIMINARIES

Consider the linear system, or “plant”

$$x_{k+1} = Ax_k + Bu_k + Ed_k, \quad (1)$$

$$y_k = Cx_k, \quad (2)$$

with state $x_k \in \mathbb{R}^n$, control $u_k \in \mathbb{R}$, process disturbance $d_k \in \mathbb{R}$, and output $y_k \in \mathbb{R}$. Assume that the initial state x_0 and disturbance d_k are realizations of zero mean Gaussian random variables X_0 and D_k , where X_0 and D_k are independent for all k , X_0 has covariance $\Sigma_{0|-1}$, and D_k is an independent identically distributed (i.i.d.) sequence with variance σ_d^2 .

The control input is based on measurements of the plant output received from a Gaussian communication channel

$$r_k = s_k + n_k. \quad (3)$$

where the channel noise n_k is a realization of an i.i.d. Gaussian random process with zero mean and variance σ_n^2 . The channel noise is also assumed to be independent of the initial state and process disturbance. Assume also that the channel input s_k must satisfy the instantaneous power constraint $\mathcal{E}\{S_k^2\} \leq \mathcal{P}$ for some specified value $\mathcal{P} > 0$.

We shall be interested in communication and control strategies for which the channel input depends causally on the plant output sequence

$$s_k = f_k(y^k), \quad (4)$$

and the control input depends causally on the sequence of channel outputs

$$u_k = g_k(r^k). \quad (5)$$

Note that the encoder (4) and the decoder (5) are potentially nonlinear and time varying. In the derivation of our results, we shall temporarily assume that the encoder has access to more information than is indicated in (4), and determine the optimal performance in this case. We then show that the same performance may be achieved without the additional information.

Denote the conditional expectation of the plant state X_{k+1} given the channel output histories $R^{k-1} = r^{k-1}$ and $R^k = r^k$ by $\hat{x}_{k|k-1} = \mathcal{E}_{r^{k-1}}\{X_k\}$ and $\hat{x}_{k|k} = \mathcal{E}_{r^k}\{X_k\}$, the associated state estimation errors by $\tilde{x}_{k|k-1} = x_k - \hat{x}_{k|k-1}$ and $\tilde{x}_{k|k} = x_k - \hat{x}_{k|k}$, and the error covariance matrices by $\Sigma_{k|k-1} = \mathcal{E}\{\tilde{x}_{k|k-1}\tilde{x}_{k|k-1}^T\}$ and $\Sigma_{k|k} = \mathcal{E}\{\tilde{x}_{k|k}\tilde{x}_{k|k}^T\}$. Similarly, denote conditional estimates of the system output by $\hat{y}_{k|k-1}$ and $\hat{y}_{k|k}$, and the conditional output estimation errors

by $\tilde{y}_{k|k-1}$ and $\tilde{y}_{k|k}$. The variance of $\tilde{y}_{k|k-1}$ is thus given by $\mathcal{E}\{\tilde{Y}_{k|k-1}^2\} = \mathcal{E}\{(Y_k - \mathcal{E}_{R^{k-1}}\{Y_k\})^2\}$, and a similar expression holds for $\mathcal{E}\{\tilde{Y}_{k|k}^2\}$.

Our problem is to choose encoding and decoding sequences $f_k(y^k)$ and $g_k(r^k)$, $k = 0, \dots, N$, to minimize the mean square value of the system output at terminal time $k = N + 1$, subject to the channel input power constraint $\mathcal{E}\{S_k^2\} \leq \mathcal{P}$. The cost function is thus given by

$$J_{N+1}^* = \inf_{\substack{f_k, g_k \\ k=0, \dots, N}} \mathcal{E}\{Y_{N+1}^2\}. \quad (6)$$

It is common to also consider a transient penalty on the system output, and to penalize the response over an infinite horizon. Although we do not present optimal solutions to such problems, in Section VII we provide a lower bound on the optimal cost together with a class of suboptimal solutions.

The fact that the conditional expectation minimizes the mean square estimation error implies, for a given choice of encoder (4) and decoder (5), that

$$\mathcal{E}\{Y_{k+1}^2\} \geq \mathcal{E}\{\tilde{Y}_{k+1|k}^2\}, \quad k = 0, \dots, N. \quad (7)$$

By choosing the control u_N appropriately, the lower bound (7) may be achieved with equality at time $N + 1$.

Lemma II.1 Assume that $CB \neq 0$. Then the control input

$$u_N = -(CB)^{-1}CA\hat{x}_{N|N} \quad (8)$$

yields

$$\hat{y}_{N+1|N} = 0, \quad y_{N+1} = \tilde{y}_{N+1|N}. \quad (9)$$

Proof: Substituting (8) into the state equations (1)-(2) yields $y_{N+1} = CAx_N - CA\hat{x}_{N|N} + CE d_N$. The assumption that the disturbance sequence D_k is zero mean and i.i.d. implies that $\hat{y}_{N+1|N} = 0$, and the fact that $y_{N+1} = \tilde{y}_{N+1|N}$ follows immediately. ■

We have shown that the optimal control u_N sets the mean square value of the plant output at time $N + 1$ equal to its theoretical minimum, which is given by the variance of the conditional estimation error. It remains to determine the encoder sequence $f_k(y^k)$, $k = 0, \dots, N$ and the decoder/controller sequence at earlier times, $g_k(r^k)$, $k = 0, \dots, N - 1$. Evidently, Lemma II.1 implies that these should be chosen to minimize the variance of the estimation error $\tilde{y}_{N+1|N}$.

Let us now consider the information that should be communicated over the channel by the encoder. Toward that end, iterate the state equations (1)-(2) to obtain

$$y_{N+1} = \left(CA^{N+1}x_0 + \sum_{j=0}^{N-1} CA^{N-j}Ed_j \right) + CE d_N + \sum_{j=0}^N CA^{N-j}Bu_j. \quad (10)$$

The disturbance d_N does not affect the plant output, and thus cannot affect the response of a causal encoder, until time $N + 1$. The control input sequence is determined by the sequence of channel outputs, and thus may be assumed to be known at the decoder. These observations imply that the only information

that needs to be communicated over the channel is the term in parentheses in (10), which we refer to as the “message”, and whose value is determined by the primitive random variables x_0 and d_0, d_1, \dots, d_{N-1} :

$$m(x_0, d^{N-1}) \triangleq CA^{N+1}x_0 + \sum_{j=0}^{N-1} CA^{N-j}Ed_j. \quad (11)$$

Although we are given $N + 1$ uses of the channel to communicate the message (11), in fact the message does not become available until the final time step. Hence at each earlier time step we may only communicate an approximation to (11). A procedure for doing so that uses linear encoding and decoding is described in Section III, wherein we simplify the problem by temporarily allowing the encoder access to the plant state, control input, and channel output. Optimality of this procedure is proven in Section IV by showing that nonlinear encoders and decoders cannot outperform the linear schemes proposed in Section III. In Section V we show that the performance of Section III can be achieved using an encoder that has access only to the state of the plant. We show in Section VI that, under appropriate additional hypotheses, the performance achieved using an encoder with access to the state of the plant may also be achieved using an encoder with access only to the plant output, as in (4).

III. AN ENCODER WITH ADDITIONAL INFORMATION

Suppose that the encoder has access to perfect measurements of the plant state, the control input, and feedback from the channel output (cf. Figure 2). We now propose a strategy for using this additional information to transmit the message (11) over the communication channel. Later, we shall show that, under appropriate hypotheses, the performance achieved with such additional information may also be achieved using an encoder of the form (4).

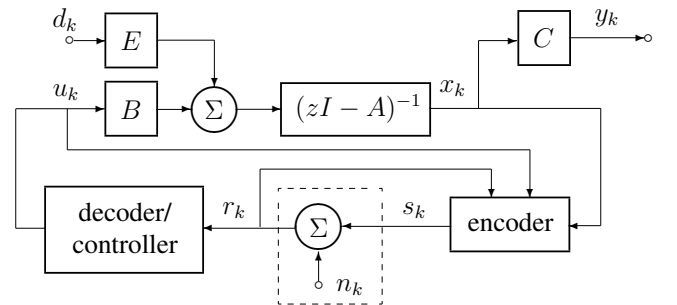


Fig. 2. An encoder that has access to the plant state, plant input, and channel output.

We first show that access to the plant state and control input allows the encoder to compute each disturbance input d_0, d_1, \dots, d_{N-1} one time step after it occurs. Choose a row vector F such that $FE \neq 0$. Then d_k may be computed at time $k + 1$:

$$d_k = (FE)^{-1}(Fx_{k+1} - FAx_k - FBu_k), \quad k = 0, \dots, N-1. \quad (12)$$

Knowledge of the primitive random variables x_0 and d_0, d_1, \dots , in turn, allows the encoder to compute an estimate of the message (11) at each time step. Denote the conditional estimate of the message (11) at time $N-1$ given the primitive random variables available at time k by $m_k = \mathcal{E}_{x_0, d^{k-1}}\{m(X_0, D^{N-1})\}$. Then an inspection of (11) reveals that m_k satisfies the recursion

$$m_{k+1} = m_k + CA^{N-k}Ed_k, \quad m_0 = CA^{N+1}x_0. \quad (13)$$

It follows that $m(x_0, d^{N-1})$ is the state of the discrete integrator (13) at time $k+1 = N$. Note that $m_{N+1} = m(x_0, d^{N-1}) + CE d_N$.

We next propose to transmit m_k over the communication channel, taking appropriate advantage of the noiseless feedback link to improve the quality of transmission, and use the sequence of channel outputs to compute an estimate of the message (11). Doing so will allow us to estimate y_{N+1} using the identity

$$y_{N+1} = m_{N+1} + \sum_{j=0}^N CA^{N-j}Bu_j. \quad (14)$$

For motivation, let us review a technique for communicating over a channel with feedback described in [4, pp. 166-168] and [15, pp. 479-481], and depicted in Figure 3. The goal in

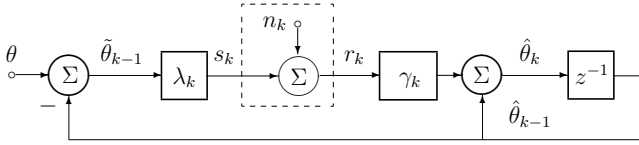


Fig. 3. Communication over a channel with noiseless feedback.

[4], [15] is to use a Gaussian channel $N+1$ times for the purpose of communicating a single message θ , assumed to be a Gaussian random variable with zero mean and variance σ_θ^2 . Denote the conditional estimate $\hat{\theta}_k \triangleq \mathcal{E}_{r^k}\{\theta\}$, $k = 0, \dots, N$, the estimation error $\tilde{\theta}_k \triangleq \theta - \hat{\theta}_k$, and set $\tilde{\theta}_{-1} = 0$. Then choosing λ_k so that $\mathcal{E}\{S_k^2\} = \mathcal{P}$, the channel power constraint, and $\gamma_k = (1/\lambda_k)(1 + \sigma_n^2/\mathcal{P})^{-1}$ results in the estimation error at time N having variance $\mathcal{E}\{\tilde{\theta}_N^2\} = \sigma_\theta^2(1 + \mathcal{P}/\sigma_n^2)^{-(N+1)}$. As described in [4], [15], rate distortion theory may be used to show that this is the minimum possible error variance, in that it cannot be further reduced through use of a more complex (e.g., nonlinear) encoder and decoder.

The scheme depicted in Figure 3 is not directly applicable to our situation, because the message (11) is not available at the beginning of channel transmission (cf. the discussion following (11)). Instead, we use the scheme depicted in Figure 4, and form an estimate of the integrator state (13) based on the output sequence of a communication channel with feedback. Note the resemblance between the estimator described below and the standard Kalman filter [1]. This resemblance is elaborated upon in Section V-A (cf. the discussion of Figure 6).

Denote the conditional expectations $\hat{m}_{k|k} \triangleq \mathcal{E}_{r^k}\{M_k\}$ and $\hat{m}_{k|k-1} \triangleq \mathcal{E}_{r^{k-1}}\{M_k\}$, and the associated estimation errors

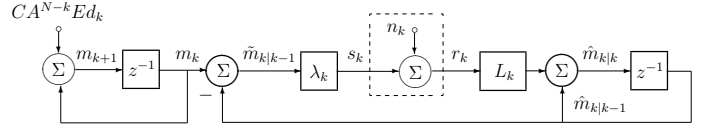


Fig. 4. Communicating the output of a discrete integrator over a channel with feedback.

by $\tilde{m}_{k|k} = m_k - \hat{m}_{k|k}$ and $\tilde{m}_{k|k-1} = m_k - \hat{m}_{k|k-1}$. The channel input and output in Figure 4 are given by

$$s_k = \lambda_k \tilde{m}_{k|k-1}, \quad r_k = s_k + n_k, \quad (15)$$

and the conditional estimates evolve according to the recursions

$$\hat{m}_{k|k} = \hat{m}_{k|k-1} + L_k r_k, \quad \hat{m}_{0|-1} = 0, \quad (16)$$

$$\hat{m}_{k+1|k} = \hat{m}_{k|k}. \quad (17)$$

The estimator gain in (16) is equal to

$$L_k = \frac{\lambda_k \mathcal{M}_{k|k-1}}{\lambda_k^2 \mathcal{M}_{k|k-1} + \sigma_n^2}, \quad (18)$$

and the error covariances $\mathcal{M}_{k|k} \triangleq \mathcal{E}\{\tilde{M}_{k|k}^2\}$ and $\mathcal{M}_{k+1|k} \triangleq \mathcal{E}\{\tilde{M}_{k+1|k}^2\}$ satisfy the Riccati equations

$$\mathcal{M}_{k|k} = \mathcal{M}_{k|k-1} - \frac{\lambda_k^2 \mathcal{M}_{k|k-1}^2}{\lambda_k^2 \mathcal{M}_{k|k-1} + \sigma_n^2}, \quad (19)$$

$$\mathcal{M}_{k+1|k} = \mathcal{M}_{k|k} + (CA^{N-k}E)^2 \sigma_d^2, \quad (20)$$

with initial condition

$$\mathcal{M}_{0|-1} \triangleq CA^{N+1}\Sigma_{0|-1}A^{(N+1)T}C^T. \quad (21)$$

It follows readily from (19)-(20) that if $\mathcal{M}_{0|-1} > 0$, then $\mathcal{M}_{k|k-1} > 0$, $\forall k \geq 0$. Similarly, if $\mathcal{M}_{0|-1} = 0$ but there exists j such that $CA^{N-j}E \neq 0$, then $\mathcal{M}_{k+1|k} > 0$, $\forall k \geq j$. We henceforth simply assume that $\mathcal{M}_{0|-1} > 0$, and at each time step choose λ_k so that

$$\lambda_k^2 \mathcal{M}_{k|k-1} = \mathcal{P}. \quad (22)$$

Then (18)-(19) reduce to

$$L_k = \frac{1}{\lambda_k} \frac{\mathcal{P}}{\mathcal{P} + \sigma_n^2}, \quad (23)$$

$$\mathcal{M}_{k|k} = \mathcal{M}_{k|k-1} \frac{\sigma_n^2}{\mathcal{P} + \sigma_n^2}. \quad (24)$$

This reduction reveals the similarity of the communication schemes in Figures 3 and 4. In each case λ_k is chosen to satisfy the channel power constraint with equality. Once λ_k is chosen, the formulas for γ_k in Figure 3 and L_k in Figure 4 are identical. The only difference is the presence of the disturbance in Figure 4. With this disturbance present, the variance of the estimation error $\tilde{m}_{N+1|N}$ may be obtained by iterating the Riccati equations (20) and (24):

$$\mathcal{M}_{N+1|N} = \mathcal{M}_{0|-1} \left(\frac{\sigma_n^2}{\mathcal{P} + \sigma_n^2} \right)^{N+1} + \sigma_d^2 \sum_{j=0}^N (CA^{N-j}E)^2 \left(\frac{\sigma_n^2}{\mathcal{P} + \sigma_n^2} \right)^{N-j}. \quad (25)$$

Were the disturbance not present in Figure 4, the estimation error would depend only on the initial state and thus be identical to that achieved using the procedure described in [4], [15].

We have shown how to use the additional information present at the encoder in Figure 4 to obtain an estimate of the message m_N with error variance (25). By the identity (14), this allows us to obtain an estimate of y_{N+1} :

$$\hat{y}_{N+1|N} = \hat{m}_{N+1|N} + \sum_{j=0}^N CA^{N-j}Bu_j. \quad (26)$$

It follows from Lemma II.1 that, under optimal control (8), the system output satisfies $y_{N+1} = \tilde{y}_{N+1|N}$, and thus from (14), (25), and (26) that the mean square value of the output is equal to

$$\begin{aligned} \mathcal{E}\{Y_{N+1}^2\} &= \mathcal{M}_{0|-1} \left(\frac{\sigma_n^2}{\mathcal{P} + \sigma_n^2} \right)^{N+1} \\ &+ \sigma_d^2 \sum_{j=0}^N (CA^{N-j}E)^2 \left(\frac{\sigma_n^2}{\mathcal{P} + \sigma_n^2} \right)^{N-j}. \end{aligned} \quad (27)$$

Note that Lemma II.1 only characterizes the control at time N in terms of the state estimate, which will depend on the previous control sequence. It follows from (12), however, that there is no loss of generality in setting the control inputs at earlier times to zero. The resulting value of u_N may then be computed from (26), and the entire control sequence is given by

$$u_k = 0, \quad k = 0, \dots, N-1, \quad u_N = -(CB)^{-1}\hat{m}_{N+1|N}. \quad (28)$$

To summarize, we have proposed linear, time varying communication and control algorithms for the purpose of minimizing $\mathcal{E}\{Y_{N+1}^2\}$ in the special case that the encoder has access to the state of the plant, the control input, and the channel output. The required computations are summarized as pseudocode in Figure 5.

IV. OPTIMALITY OF LINEAR TIME-VARYING COMMUNICATION AND CONTROL

We now show that the communication and control strategies proposed in Section III are optimal for encoders with the specified additional information, in the sense that no other causal strategies can achieve a smaller value of $\mathcal{E}\{Y_{N+1}^2\}$. As noted in Section III, rate distortion theory [4], [15] may be used to show that the corresponding scheme with no disturbance present is optimal. In general, one would not expect linear strategies to be optimal for communication constrained control problems, although they may indeed be optimal for scalar systems [3]. The latter observation is relevant to the present discussion because, as we have shown, giving the encoder access to additional information reduces the optimal control problem (6) to that of estimating the state of a scalar dynamical system, specifically that of the discrete integrator (13).

Our approach to proving optimality is based on the results of [14], wherein the encoder and decoder were assumed to have the form (4)-(5), and a lower bound was established on the

begin Initialization

$$m_0 = CA^{N+1}x_0; \quad \% (13)$$

$$\hat{m}_{0|-1} = 0; \quad \% (16)$$

$$\mathcal{M}_{0|-1} = CA^{N+1}\Sigma_{0|-1}A^{(N+1)T}C^T; \quad \% (21)$$

$$u_0 = 0; \quad \% (28)$$

end

for $k = 0$ **to** N **do**

$$d_k = (FE)^{-1}(Fx_{k+1} - FAx_k - FBu_k); \quad \% (12)$$

$$m_{k+1} = m_k + CA^{N-k}Ed_k; \quad \% (13)$$

$$s_k = \lambda_k \tilde{m}_{k|k-1}, \quad \mathcal{E}\{S_k^2\} = \mathcal{P}; \quad \% (15), (22)$$

$$\hat{m}_{k|k} = \hat{m}_{k|k-1} + \frac{1}{\lambda_k} \frac{\mathcal{P}}{\mathcal{P} + \sigma_n^2} r_k; \quad \% (16), (23)$$

$$\hat{m}_{k+1|k} = \hat{m}_{k|k}; \quad \% (17)$$

$$\mathcal{M}_{k|k} = \mathcal{M}_{k|k-1} \frac{\sigma_n^2}{\mathcal{P} + \sigma_n^2}; \quad \% (24)$$

$$\mathcal{M}_{k+1|k} = \mathcal{M}_{k|k} + (CA^{N-k}E)^2 \sigma_d^2; \quad \% (20)$$

$$\text{if } k < N \text{ then} \quad \% (28)$$

$$| \quad u_k = 0;$$

else

$$| \quad u_N = -(CB)^{-1}\hat{m}_{N+1|N};$$

Fig. 5. Pseudocode: communication and control algorithms for an encoder that computes the disturbance using access to the plant state, control input, and channel output.

conditional entropy power of the state of a linear dynamical system given a sequence of outputs from a Gaussian channel. Conditional entropy power provides a *lower bound* on the variance of the conditional estimation error [9, p. 255] and thus, by inequality (7), on the mean square value of the system output. It was shown in [14] that for a scalar dynamical system this bound is tight, and may be achieved using linear encoding and decoding.

Consider the scalar linear system (13) with state m_k . Following [14], define the conditional entropy of M_k given the output sequence $R^k = r^k$ by $H_{r^k}(M_k) = -\mathcal{E}\{\log_e p_{M_k|r^k}(M_k)\}$, the conditional entropy power of M_k given $R^k = r^k$ by $N_{r^k}(M_k) = (1/2\pi e)e^{2H_{r^k}(M_k)}$, and the average conditional entropy power of M_k given $R^k = r^k$ and averaged over R^k by $N(M_k|R^k) = \mathcal{E}\{N_{R^k}(M_k)\}$. For simplicity, let $n_{k|k} \triangleq N(M_k|R^k)$, and define $n_{k+1|k} \triangleq N(M_{k+1}|R^k)$.

It follows by applying Lemmas III.2-III.3 of [14] to the system (13) that

$$n_{k|k} \geq n_{k|k-1} \left(\frac{\sigma_n^2}{\mathcal{P} + \sigma_n^2} \right), \quad (29)$$

$$n_{k+1|k} \geq n_{k|k} + (CA^{N-k}E)^2 \sigma_d^2. \quad (30)$$

Iterating the recursions (29)-(30) yields a lower bound on $n_{N+1|N}$, the average conditional entropy power of M_{N+1} given the channel output sequence R^N . The fact that conditional entropy power provides a lower bound on the variance of the conditional estimation error [9], [14], [25] implies that

$\mathcal{M}_{N+1|N} \geq n_{N+1|N}$, and thus

$$\mathcal{E}\{\tilde{Y}_{N+1|N}^2\} \geq \left(\frac{\sigma_n^2}{\mathcal{P} + \sigma_n^2}\right)^{N+1} n_{0|-1} + \sigma_d^2 \sum_{j=0}^N (CA^{N-j}E)^2 \left(\frac{\sigma_n^2}{\mathcal{P} + \sigma_n^2}\right)^{N-j}, \quad (31)$$

where $n_{0|-1} = CA^{N+1}\Sigma_{0|-1}A^{(N+1)T}C^T$. Note for later reference that we may obtain a lower bound on $\mathcal{E}\{\tilde{Y}_{k+1|k}^2\}$ by replacing each occurrence of N in (31) with k .

The lower bound on conditional estimation error (31) holds *only* under the assumption that the estimation is conditioned on $N+1$ outputs of a Gaussian channel, and *does not* assume that the encoder or decoder/controller is linear. Together, the lower bounds (7) and (31) yield a theoretical lower bound on the mean square value of the output y_{N+1} . Comparing this lower bound with (27) shows that the linear communication and control strategies developed in Section III cannot be outperformed by more general, possibly nonlinear, strategies. The proof of optimality uses in an essential way the fact that the problem in Section III has been reduced to one of estimating the state of a scalar dynamical system. The authors of [20, Section 16.4.5] show that a linear encoder and decoder are optimal for the problem of transmitting the state of a scalar linear system over a continuous-time Gaussian channel. For discrete-time systems, the authors of [3] show that linear strategies are optimal for certain scalar communication and control problems. Given these observations, the present results may not be surprising.

V. AN ENCODER WITH STATE INFORMATION ONLY

It is not realistic to assume, as was done in Section III, that the encoder has access to the channel output, plant state, and control input. However, we may show that under appropriate additional hypotheses it is possible to achieve the same performance as in Section III using an encoder that has access to only the plant output. Doing so requires several steps, which we undertake in the present section and in Section VI.

We first assume in Section V-A that the encoder retains access to all the information available in Section III, but rather than explicitly computing the disturbance input sequence, instead transmits a linear combination of the plant states over the channel. Although this communication scheme retains the same information pattern at the encoder as that used in Section III, we shall see in Section V-B that it is easier to generalize to an encoder that has access to less information. By way of contrast, the scheme of Section III is more useful for the purpose of proving optimality, as it reduces the problem to one for a scalar system.

Recall that the control input used in Section III was assumed equal to zero until the final time step. This assumption is maintained, with no loss of generality, in Section V-A. Assuming that the control is zero until the final time step is no longer possible in Section V-B, wherein we show that an appropriate choice of the control input allows the noiseless feedback path around the channel to be replaced by feedback

through the plant. The assumption that the encoder has access to the plant states is removed in Section VI.

The authors of [7] consider the problem of estimation over a Gaussian channel with feedback from the channel output to the encoder, as we do in Section V-A. Using rate distortion theory, they determine an encoder and decoder that use the channel optimally (see also [30]). They also consider an infinite horizon optimal control problem and show that a separation property exists between the problems of communication and control provided that the channel has noiseless feedback from the output to the encoder. In Section V-B, on the other hand, we consider a finite horizon control problem with only a terminal cost, and replace the noiseless feedback from the channel output to the encoder with an appropriately chosen control signal. Hence, in our scenario, the separation property shown in [7] is no longer present.

A. Stage 1: Encoder Retains Access to the Channel Output and the Control Input

It follows from Lemma II.1 that if the control signal u_N satisfies (8), then the problem of minimizing the terminal cost reduces to one of estimation, and setting $u_k = 0$, $k = 0, \dots, N-1$, results in a problem of estimating the state of an uncontrolled plant over a channel with feedback. Our approach to this problem is depicted in Figure 6, wherein we define a time-varying linear combination of states

$$z_k \triangleq H_k x_k, \quad (32)$$

and consider the channel input

$$s_k = \lambda_k \tilde{z}_{k|k-1}, \quad (33)$$

where $\hat{z}_{k|k-1} = \mathcal{E}\{Z_k | r^{k-1}\}$ and $\tilde{z}_{k|k-1} = z_k - \hat{z}_{k|k-1}$. For given sequences H_k and λ_k , the conditional state estimate satisfies the recursion

$$\hat{x}_{k+1|k} = A\hat{x}_{k|k-1} + AL_k r_k, \quad (34)$$

with initial condition $\hat{x}_{0|-1} = 0$. The sequences of estimator gains L_k and error covariance matrices $\Sigma_{k+1|k}$ are given by

$$L_k = \lambda_k \Sigma_{k|k-1} H_k^T / (\lambda_k^2 H_k \Sigma_{k|k-1} H_k^T + \sigma_n^2), \quad (35)$$

$$\Sigma_{k+1|k} = A \Sigma_{k|k-1} A^T - \frac{\lambda_k^2 A \Sigma_{k|k-1} H_k^T H_k \Sigma_{k|k-1} A^T}{\lambda_k^2 H_k \Sigma_{k|k-1} H_k^T + \sigma_n^2} + \sigma_d^2 E E^T, \quad (36)$$

with initial condition $\Sigma_{0|-1}$. For later reference, we note that the state estimation error satisfies

$$\tilde{x}_{k+1|k} = A\tilde{x}_{k|k-1} + Ed_k - AL_k r_k, \quad (37)$$

with $\tilde{x}_{0|-1} = x_0$.

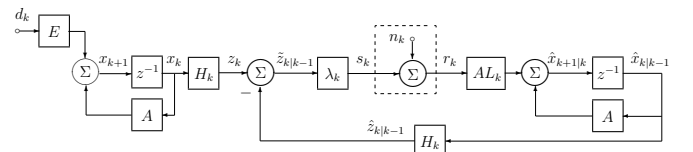


Fig. 6. Estimation over a channel with feedback.

Note the resemblance between the communication scheme in Figure 6 and a standard predicting state estimator [1]. Indeed, it is possible to transform the former into the latter by absorbing λ_k into H_k and moving the feedback summing junction from the input to the output of the communication channel. These manipulations do not affect the response of the state estimation error $\hat{x}_{k+1|k}$ to the disturbance and noise inputs and to initial conditions, and thus the expressions (35) and (36) for the estimator gain and error covariance are identical to those for a predicting estimator. This equivalence yields interesting interpretations; for example, the channel output in Figure 6 is identical to the innovations sequence of an optimal estimator, and is thus a white noise sequence [1].

We now show how to choose the sequences H_k and λ_k , $k = 0, \dots, N$, so that the estimation error at time $N + 1$ satisfies the lower bound (31) with equality. For later reference, note that the covariance of the output estimation error $\tilde{y}_{k+1|k}$ satisfies

$$C\Sigma_{k+1|k}C^T = C A \Sigma_{k|k-1} A^T C^T - \frac{\lambda_k^2 C A \Sigma_{k|k-1} H_k^T H_k \Sigma_{k|k-1} A^T C^T}{\lambda_k^2 H_k \Sigma_{k|k-1} H_k^T + \sigma_n^2} + \sigma_d^2 (CE)^2. \quad (38)$$

Our first step is to choose the final values of the sequences, H_N and λ_N .

Lemma V.1 Assume that $C A \Sigma_{N|N-1} A^T C^T > 0$. Then values of H_N and λ_N that minimize $\mathcal{E}\{\tilde{Y}_{N+1|N}^2\}$, subject to the power constraint $\mathcal{E}\{S_N^2\} \leq \mathcal{P}$, are given by $H_N = CA$ and $\lambda_N^2 = \mathcal{P}/H_N \Sigma_{N|N-1} H_N^T$. Furthermore, with these choices of H_N and λ_N ,

$$\mathcal{E}\{\tilde{Y}_{N+1|N}^2\} = C A \Sigma_{N|N-1} A^T C^T \left(\frac{\sigma_n^2}{\mathcal{P} + \sigma_n^2} \right) + \sigma_d^2 (CE)^2. \quad (39)$$

Proof: Any positive semidefinite matrix $X \in \mathbb{R}^{n \times n}$, whose rank is equal to m , has a matrix square root $Y \in \mathbb{R}^{n \times m}$ with rank m that satisfies $X = YY^T$. Denote such a square root for $\Sigma_{N|N-1}$ by Y_N . It follows that

$$C A \Sigma_{N|N-1} H_N^T = \|C A Y_N\| \|H_N Y_N\| \cos \phi_N, \quad (40)$$

where $\|\cdot\|$ denotes the Euclidean vector norm, and $\cos \phi_N \triangleq |C A Y_N Y_N^T H_N^T| / (\|C A Y_N\| \|H_N Y_N\|)$. Substituting (40) into (38) and rearranging yields

$$C \Sigma_{N+1|N} C^T = \sigma_d^2 (CE)^2 + C A \Sigma_{N|N-1} A^T C^T \left(\frac{\lambda_N^2 H_N \Sigma_{N|N-1} H_N^T \sin^2 \phi_N + \sigma_n^2}{\lambda_N^2 H_N \Sigma_{N|N-1} H_N^T + \sigma_n^2} \right). \quad (41)$$

It is straightforward to show that the coefficient of $C A \Sigma_{N|N-1} A^T C^T$ in (41) is a monotonically decreasing function of λ_N^2 , and thus λ_N should be chosen to satisfy the power constraint with equality. Doing so yields

$$C \Sigma_{N+1|N} C^T = \sigma_d^2 (CE)^2 + C A \Sigma_{N|N-1} A^T C^T \left(\frac{\mathcal{P} \sin^2 \phi_N + \sigma_n^2}{\mathcal{P} + \sigma_n^2} \right).$$

Since we assume that $\Sigma_{N|N-1}$ is given, it follows that H_N should be chosen as a scalar multiple of CA , in which case $\phi_N = 0$. ■

Our next result builds on Lemma V.1 to exhibit choices of H_k and λ_k such that the variance of the output estimation error at time $N + 1$ achieves the lower bound (31).

Proposition V.2 Consider the communication channel with feedback depicted in Figure 6. Choose the channel input s_k , $k = 0, \dots, N$, to satisfy (33), where

$$H_k \triangleq C A^{N+1-k}. \quad (42)$$

Assume that $H_k \Sigma_{k|k-1} H_k^T > 0$, and choose λ_k such that

$$\lambda_k^2 = \mathcal{P} / H_k \Sigma_{k|k-1} H_k^T. \quad (43)$$

Then the variance of the estimation error at time $k = N + 1$ satisfies

$$\mathcal{E}\{\tilde{Y}_{N+1|N}^2\} = Z_{0|-1} \left(\frac{\sigma_n^2}{\mathcal{P} + \sigma_n^2} \right)^{N+1} + \sigma_d^2 \sum_{j=0}^N (C A^{N-j} E)^2 \left(\frac{\sigma_n^2}{\mathcal{P} + \sigma_n^2} \right)^{N-j}, \quad (44)$$

where $Z_{0|-1} \triangleq \mathcal{E}\{\tilde{Z}_{0|-1}^2\} = C A^{N+1} \Sigma_{0|-1} A^{(N+1)T} C^T$.

Proof: We have shown in Lemma V.1 that the choices of H_N and λ_N given by (42) and (43) minimize the estimation error for a given value of $\Sigma_{N|N-1}$. It follows from (39) that the problem of choosing H_{N-1} and λ_{N-1} to minimize $C \Sigma_{N+1|N} C^T$ reduces to that of minimizing $C A \Sigma_{N|N-1} A^T C^T$ for a given value of $\Sigma_{N-1|N-2}$. Computations similar to those in the proof of Lemma V.1 show that $C A \Sigma_{N|N-1} A^T C^T$ is minimized using the values of H_{N-1} and λ_{N-1} given by the formulas (42)-(43) with $k = N - 1$. Repeating this process yields (44). ■

Remark V.3 The assumption that $H_k \Sigma_{k|k-1} H_k^T > 0$ in Proposition V.2 is not restrictive. Indeed, it is straightforward to show that

$$H_{k+1} \Sigma_{k+1|k} H_{k+1}^T = H_k \Sigma_{k|k-1} H_k^T \frac{\sigma_n^2}{\mathcal{P} + \sigma_n^2} + \sigma_d^2 (H_{k+1} E)^2,$$

and thus if $H_0 \Sigma_{0|-1} H_0^T > 0$, then $H_k \Sigma_{k|k-1} H_k^T > 0$, $\forall k \geq 0$. Similarly, if $H_j E \neq 0$ then $H_k \Sigma_{k|k-1} H_k^T > 0$, $\forall k \geq j$. ■

Corollary V.4 Consider the feedback system of Figure 2, assume that $u_k = 0$, $k = 0, \dots, N - 1$, and that u_N satisfies (8). Then $y_{N+1} = \tilde{y}_{N+1|N}$, and the value of $\mathcal{E}\{Y_{N+1}^2\}$ is given by (44). ■

Let us now provide an interpretation to the linear combination of states $z_k = H_k x_k$, where H_k is defined by (42). To do so, we iterate the state equations (1)-(2), yielding

$$y_{N+1} = C A^{N+1-k} x_k + \sum_{j=k}^N C A^{N-j} B u_j + \sum_{j=k}^N C A^{N-j} E d_j. \quad (45)$$

Denote the output at the terminal time under *no control* ($u_j = 0, j = 0, \dots, N$) by y_{N+1}^{NC} . Then z_k is equal to the conditional expectation of Y_{N+1}^{NC} given the state at time k : $z_k = \mathcal{E}\{Y_{N+1}^{NC} | x_k\}$. Since the values of z_k at different times are correlated with one another, it follows as in Section III that optimal use of the channel requires feedback from the channel output to make the channel input proportional to $\tilde{z}_{k|k-1}$. In Section V-B, we shall see how the feedback from the channel output may be replaced by an appropriately chosen control input. By way of contrast, in the present section the control input is chosen as in Corollary V.4.

B. Stage 2: Encoder has Access only to the Plant State

Suppose that the encoder has access only to the state of the plant. We now show that by using a specific choice of the control input sequence, state information alone is sufficient to yield performance identical to that achieved by the scheme in Section V-A. A block diagram of the resulting communication and control system is depicted in Figure 7.

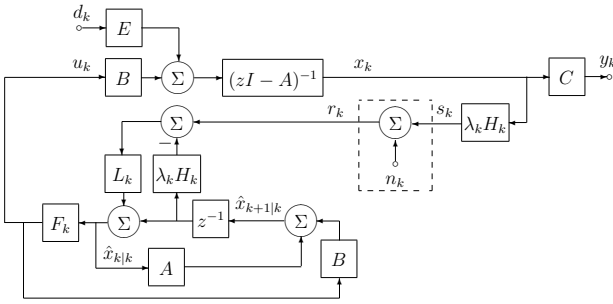


Fig. 7. Communication and control with an encoder that has access only to the plant state.

Proposition V.5 Consider the plant (1)-(2) and the communication channel (3). Define the sequence H_k as in (42), and assume that both $H_k \Sigma_{k|k-1} H_k^T > 0$ and $H_{k+1} B \neq 0$, for $k = 0, \dots, N$. Let the channel input sequence be given by

$$s_k = \lambda_k z_k, \quad z_k = H_k x_k, \quad k = 0, \dots, N, \quad (46)$$

where λ_k is chosen as in (43). Choose the control sequence

$$u_k = -F_k \hat{x}_{k|k}, \quad (47)$$

where the control gain is defined by

$$F_k \triangleq (H_{k+1} B)^{-1} H_{k+1} A, \quad (48)$$

and the state estimate $\hat{x}_{k|k}$ satisfies

$$\hat{x}_{k|k} = \hat{x}_{k|k-1} + L_k (r_k - \lambda_k H_k \hat{x}_{k|k-1}), \quad (49)$$

$$\hat{x}_{k+1|k} = A \hat{x}_{k|k} + B u_k, \quad (50)$$

with initial condition $\hat{x}_{0|-1} = 0$ and estimator gain

$$L_k = \frac{\lambda_k \Sigma_{k|k-1} H_k^T}{\mathcal{P} + \sigma_n^2}, \quad (51)$$

$$\Sigma_{k+1|k} = A \Sigma_{k|k-1} A^T - \frac{\lambda_k^2 A \Sigma_{k|k-1} H_k^T H_k \Sigma_{k|k-1} A^T}{\mathcal{P} + \sigma_n^2} + \sigma_d^2 E E^T. \quad (52)$$

Then $\mathcal{E}\{Y_{N+1}^2\} = \mathcal{E}\{\tilde{Y}_{N+1|N}^2\}$, where $\mathcal{E}\{\tilde{Y}_{N+1|N}^2\}$ is given by (44).

Proof: The assumption that X_0 is a zero mean random variable implies that $\hat{x}_{0|-1} = 0$, and thus that $z_0 = \tilde{z}_{0|-1}$. Furthermore, the definition of z_k , the state equation (1), and the identity $H_{k+1} A = H_k$ imply that $z_{k+1} = H_k x_k + H_{k+1} B u_k + H_{k+1} E d_k$, and substituting u_k yields $z_{k+1} = H_k \hat{x}_{k|k} + H_{k+1} E d_k$. Since $\tilde{X}_{k|k}$ and D_k are zero mean random variables, it follows that $\tilde{z}_{k+1|k} = 0$ and $z_{k+1} = \tilde{z}_{k+1|k}$, for $k = 0, \dots, N$. The latter facts have two implications. First, the definition of z_k implies that $H_k \hat{x}_{k|k-1} = 0$, and thus that (49)-(50) reduce to (37). Second, the channel input sequence s_k defined by (46) is identical to the channel input sequence (33) used in Proposition V.2, and thus the two channel output sequences r_k are also identical. Since the two state estimation error sequences both satisfy the difference equation (37) with the same initial condition and the same inputs r_k and d_k , it follows that their values are identical at each time step. Finally, (42) implies that $H_{N+1} = C$, and thus u_N defined by (47)-(48) is equal to u_N from Lemma II.1, and $y_{N+1} = \tilde{y}_{N+1|N}$, where $\tilde{y}_{N+1|N}$ has variance (44). ■

Let us compare the result of Proposition V.5 with that of Section V-A, wherein the encoder had access to the *channel output*. Such access enabled the encoder to transmit the estimation error $\tilde{z}_{k|k-1}$ over the channel. The control input played no role in estimation, and was assumed to be zero until the final time step, after estimation had been completed. An encoder with access only to the *state* of the plant cannot compute the estimation error $\tilde{z}_{k|k-1}$. However, the control defined in (47) sets $\hat{z}_{k|k-1} = 0$, and thus the plant output is equal to the estimation error: $z_k = \tilde{z}_{k|k-1}$. For $k = 0, \dots, N$, this implies that the channel input s_k defined by (46) is identical to that in Proposition V.2. At the terminal time $k = N + 1$, the fact that $H_{N+1} = C$ implies that the value of y_{N+1} is the same as that obtained in Corollary V.4.

We now provide an additional interpretation to the optimal control input. Substituting u_k defined by (47)-(48) into (45) yields

$$y_{N+1} = C A^{N+1-k} \hat{x}_{k|k} + \sum_{j=k+1}^N C A^{N-j} B u_j + \sum_{j=k}^N C A^{N-j} E d_j. \quad (53)$$

Suppose that future values of the control input are set equal to zero: $u_j = 0, j = k + 1, \dots, N$. Then (53) implies that the optimal control at time k sets the conditional expectation of the plant output at time $N + 1$ equal to zero: $\mathcal{E}\{Y_{N+1} | r^k\} = 0$. It is also interesting to consider the *state feedback* control law $u_k = -F_k x_k$, where F_k is defined by (48). This control law minimizes the cost function $J_z \triangleq \sum_{k=0}^{N+1} \mathcal{E}\{Z_k^2\}$, where z_k is defined by (46). To show this, we note that the optimal state feedback has the form [5] $F_k = -(B^T P_{k+1} B)^{-1} B^T P_{k+1} A$, where P_k is the solution to the Riccati difference equation

$$P_k = A^T P_{k+1} A - A^T P_{k+1} B (B^T P_{k+1} B)^{-1} B^T P_{k+1} A + H_k^T H_k,$$

with terminal constraint $P_{N+1} = H_{N+1}^T H_{N+1}$. It is easy to verify by backwards induction from $k = N + 1$ to $k = 0$ that $P_k = H_k^T H_k$ and $F_k = (H_{k+1} B)^{-1} H_{k+1} A$, thus

agreeing with (48). Furthermore, the control input satisfies $u_k = -(CA^{N-k}B)^{-1}CA^{N+1-k}Ed_{k-1}$ and thus, at each time step k , cancels the effect upon y_{N+1} of the disturbance at the previous time step.

Only in special cases will the optimal control at time k set the conditional expectation of y_{k+1} equal to zero. One of these occurs when the plant is first order ($n = 1$ in (1)). In this case, the control gain (48) reduces to $F_k = A/B$, and is thus *independent* of the terminal time $N + 1$. More generally, all that (7) tells us about the mean square output at transient times is that it is bounded below by the variance of the conditional estimation error. We now derive a general formula for transient values of the mean square plant output when the communication and control strategies (46) and (47) are applied.

Corollary V.6 Assume that the hypotheses of Proposition V.5 are satisfied. Then

$$\mathcal{E}\{Y_{k+1}^2\} = C\Sigma_{k+1|k}C^T + C(A - BF_k)\Gamma_{k|k}(A - BF_k)^TC^T, \quad (54)$$

where F_k is defined by (48), and $\Gamma_{k|k} \triangleq \mathcal{E}\{\hat{X}_{k|k}\hat{X}_{k|k}^T\}$ satisfies the recursion

$$\Gamma_{k+1|k+1} = (A - BF_k)\Gamma_{k|k}(A - BF_k)^T + \frac{\lambda_k^2 \Sigma_{k|k-1} H_k^T H_k \Sigma_{k|k-1}}{\mathcal{P} + \sigma_n^2}, \quad (55)$$

with initial condition $\Gamma_{0|0} = \lambda_0^2 \Sigma_{0|-1} H_0^T H_0 \Sigma_{0|-1} / (\mathcal{P} + \sigma_n^2)$.

Proof: Substituting the control (47) into the state equation (1) and applying the definition of $\tilde{x}_{k|k}$ yields $y_{k+1} = CA\tilde{x}_{k|k} + C(A - BF_k)\hat{x}_{k|k} + CE d_k$. It follows from the orthogonal projection principle [5, Appendix E] that $\mathcal{E}\{\tilde{x}_{k|k}\hat{x}_{k|k}^T\} = 0$. This fact, together with the assumption that d_k is i.i.d., implies that

$$\mathcal{E}\{Y_{k+1}^2\} = \mathcal{E}\{(CA\tilde{x}_{k|k})^2\} + \mathcal{E}\{(C(A - BF_k)\hat{x}_{k|k})^2\} + (CE)^2\sigma_d^2. \quad (56)$$

The fact that $\tilde{x}_{k+1|k} = A\tilde{x}_{k|k} + Ed_k$ implies that the sum of the first and third terms on the right hand side of (56) is equal to $C\Sigma_{k+1|k}C^T$, and applying the definition of $\Gamma_{k|k}$ yields (54). The conditional estimate $\hat{x}_{k|k}$ satisfies the recursion $\hat{x}_{k|k} = \hat{x}_{k|k-1} + L_k(\lambda_k H_k \tilde{x}_{k|k-1} + n_k)$, where λ_k and L_k are chosen as in (43) and (35). The orthogonal projection principle thus implies $\Gamma_{k|k} = \mathcal{E}\{\hat{X}_{k|k-1}\hat{X}_{k|k-1}^T\} + L_k L_k^T (\mathcal{P} + \sigma_n^2)$. Noting that $\hat{x}_{k|k-1} = A\hat{x}_{k-1|k-1} + Bu_{k-1} = (A - BF_{k-1})\hat{x}_{k-1|k-1}$ yields $\mathcal{E}\{\hat{X}_{k|k-1}\hat{X}_{k|k-1}^T\} = (A - BF_{k-1})\Gamma_{k-1|k-1}(A - BF_{k-1})^T$, and substituting for L_k from (35) yields (55). The expression for $\Gamma_{0|0}$ follows since $\hat{x}_{0|-1} = 0$. ■

In the scalar case, it is easy to verify that $A - BF_k = 0$, and thus that (7) is satisfied with equality at each time step. More generally, the second term on the right hand side of (54) quantifies precisely the additional cost produced by this control law at every time $k < N + 1$.

We next present an example to illustrate the transient behavior of the output and output estimation error for a communication and control strategy designed according to

Proposition V.5. For purposes of comparison, we also plot a theoretical lower bound on the variance of the output estimation error at transient times. To do so, we recall that the entropy power arguments used to derive (31) show that at each time k , the estimation error is bounded below by

$$\mathcal{E}\{\tilde{Y}_{k+1|k}^2\} \geq \left(\frac{\sigma_n^2}{\mathcal{P} + \sigma_n^2}\right)^{k+1} n_{0|-1} + \sigma_d^2 \sum_{j=0}^k (CA^{k-j}E)^2 \left(\frac{\sigma_n^2}{\mathcal{P} + \sigma_n^2}\right)^{k-j}, \quad (57)$$

where $n_{0|-1} = CA^{k+1}\Sigma_{0|-1}A^{(k+1)T}C^T$.

Example V.7 Consider the system (1)-(2) with

$$A = \begin{bmatrix} 1.1 & 1 \\ 0 & 1.2 \end{bmatrix}, \quad B = E = \begin{bmatrix} 0 \\ 1.5 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 1 \end{bmatrix},$$

$$\sigma_d^2 = 1, \quad \Sigma_{0|-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad (58)$$

and the channel (3) with $\mathcal{P} = 10$ and $\sigma_n^2 = 5$. Suppose we apply the optimal communication and control sequences (46) and (47) that minimize the cost (6) at time $N + 1 = 21$. The transient value of the estimation error variance $\mathcal{E}\{\tilde{Y}_{k+1|k}^2\}$ may be computed from (38) with H_k and λ_k given by (42) and (43). The transient value of $\mathcal{E}\{Y_{k+1}^2\}$ may be computed from (54). Both these sequences are plotted in Figure 8. Note that $\mathcal{E}\{Y_{k+1}^2\} \geq \mathcal{E}\{\tilde{Y}_{k+1|k}^2\}$, $\forall k$, as we expect from (7), and that equality is achieved at the terminal time $N + 1 = 21$. Also plotted in Figure 8 is the lower bound (57). As predicted by the theory, the value of $\mathcal{E}\{\tilde{Y}_{k+1|k}^2\}$, and thus that of $\mathcal{E}\{Y_{k+1}^2\}$, is equal to this lower bound at the terminal time. ■

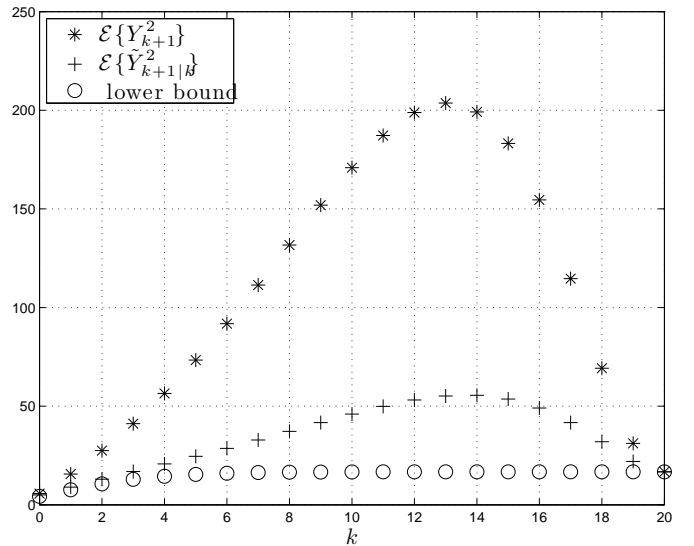


Fig. 8. Plots of $\mathcal{E}\{Y_{k+1}^2\}$ and $\mathcal{E}\{\tilde{Y}_{k+1|k}^2\}$ vs. k for terminal time $N + 1 = 21$ using the communication and control strategies (46)-(47). Also plotted is the lower bound (57) on $\mathcal{E}\{\tilde{Y}_{k+1|k}^2\}$.

It is interesting to note that there is no “separation” between the tasks of control and estimation at the decoder. Both the

control gain F_k defined by (48) and the estimator gain L_k defined by (35) depend on the linear combination H_k of states to be transmitted over the channel. The control law applied at times $k = 0, \dots, N-1$ is designed to aid in computing the optimal estimate $\hat{y}_{N+1|N}$, and the control law at time $k = N$ sets the output equal to the optimal estimation error. Hence, although there is no separation between control and estimation, there is also no conflict because the sole purpose of control is to aid in estimation until the final step, at which time estimation relevant to terminal performance has been completed.

VI. AN ENCODER WITH OUTPUT INFORMATION ONLY

We assumed in Section III that the encoder had access to the plant state, the control input, and the channel output, and in Section V-B we assumed access only to the plant state. We now replace this assumption by allowing the encoder access only to the response of a linear filter to a noiseless measurement of the plant output. Under appropriate additional hypotheses, the same performance will be achieved as when the encoder had access to the plant state. As we shall see, the filter has the form of a state estimator that *does not* require knowledge of the control input to the plant.

We shall consider the linear system (1)-(2), and construct an optimal estimate for the state given noise-free measurements of the plant output and control input. To distinguish these state estimates from those obtained by processing the channel output, we define the conditional expectations $\hat{x}_{k+1|k}^0 \triangleq \mathcal{E}\{X_{k+1}|u^k, y^k\}$ and $\hat{x}_{k|k}^0 \triangleq \mathcal{E}\{X_k|u^{k-1}, y^k\}$, and denote the associated error covariance matrices by $\Sigma_{k+1|k}^0$ and $\Sigma_{k|k}^0$, respectively. The conditional state estimates $\hat{x}_{k+1|k}^0$ and $\hat{x}_{k|k}^0$ satisfy the recursions

$$\hat{x}_{k+1|k}^0 = A\hat{x}_{k|k}^0 + Bu_k \quad (59)$$

$$\hat{x}_{k|k}^0 = \hat{x}_{k|k-1}^0 + L_k^0(y_k - C\hat{x}_{k|k-1}^0), \quad (60)$$

where $L_k^0 = \Sigma_{k|k-1}^0 C^T (C\Sigma_{k|k-1}^0 C^T)^{-1}$ and

$$\Sigma_{k+1|k}^0 = A\Sigma_{k|k}^0 A^T + \sigma_d^2 EE^T \quad (61)$$

$$\Sigma_{k|k}^0 = \Sigma_{k|k-1}^0 - \Sigma_{k|k-1}^0 C^T (C\Sigma_{k|k-1}^0 C^T)^{-1} C\Sigma_{k|k-1}^0. \quad (62)$$

Under appropriate hypotheses, the solution to (61)-(62) converges to a steady state solution with special properties. Denote the transfer function from d_k to y_k in (1)-(2) by $G_d(z) = C\Phi(z)E$.

Lemma VI.1 Assume that (A, E) is stabilizable, (A, C) is detectable, and that $G_d(z)$ has relative degree one and is strictly minimum phase. Then, as $k \rightarrow \infty$, $\Sigma_{k+1|k}^0 \rightarrow \Sigma^{ss}$ and $L_k^0 \rightarrow L^{ss}$, where

$$\Sigma^{ss} = \sigma_d^2 EE^T, \quad L^{ss} = E(CE)^{-1}, \quad (63)$$

and $\lim_{k \rightarrow \infty} \Sigma_{k|k} = 0$. Furthermore, the eigenvalues of $A - AL^{ss}C$ are stable and lie at the $n-1$ zeros of $G_d(z)$ and the origin.

Proof: The fact that stabilizability and detectability imply convergence is a standard result [1], and the assumption of

relative degree one implies $CE \neq 0$. The special form that Σ^{ss} and L^{ss} take follows from the discussion in [27, Theorem 3.3], and the locations of the closed loop eigenvalues are given by the dual of [19, Theorem 6.37(b)]. The fact that $\Sigma_{k|k}$ converges to zero follows by substituting Σ^{ss} into (62). ■

In general it will be necessary for the encoder to have access to the control input in order to implement the estimator (59)-(60). Suppose, however, that we implement a suboptimal estimator for x_k obtained by combining (59) and (60) and using L^{ss} instead of the optimal gain L_k^0 . Denoting the state of this estimator by $\hat{x}_{k|k}^s$, we have that

$$\hat{x}_{k+1|k+1}^s = A\hat{x}_{k|k}^s + Bu_k + L^{ss}(y_{k+1} - CA\hat{x}_{k|k}^s - CBu_k). \quad (64)$$

We now show, under one additional hypothesis, that the state estimate in (64) does not depend on the control input, and converges asymptotically to the state of the plant (1).

Proposition VI.2 Assume that the disturbance in (1) enters at the control input ($E = B$) and that the hypotheses of Lemma VI.1 hold. Then the state estimate $\hat{x}_{k|k}^s$ defined by (64) may be obtained from

$$\hat{x}_{k+1|k+1}^s = A\hat{x}_{k|k}^s + L^{ss}(y_{k+1} - CA\hat{x}_{k|k}^s), \quad (65)$$

where L^{ss} is given by (63) and $\hat{x}_{0|0}^s = \hat{x}_{0|-1}^s + L^{ss}(y_0 - C\hat{x}_{0|-1}^s)$. Furthermore, the state estimation error $\tilde{x}_{k|k}^s$ satisfies

$$\tilde{x}_{k+1|k+1}^s = (A - L^{ss}CA)\tilde{x}_{k|k}^s, \quad (66)$$

where the eigenvalues of $A - L^{ss}CA$ lie at the $n-1$ zeros of $G_d(z)$ and the origin.

Proof: Substituting (63) into (64) yields (65), and subtracting (65) from (1) yields (66). The eigenvalues of $A - L^{ss}CA$ are identical to those of $A - AL^{ss}C$ [1, p. 331] given by Lemma VI.1. ■

Proposition VI.2 implies that if the estimator is initialized so that $\hat{x}_{0|-1}^s = x_0$, then $\hat{x}_{k|k}^s = x_k$ for $k = 0, 1, \dots$. Otherwise $\hat{x}_{k|k}^s \rightarrow x_k$ at a rate determined by the eigenvalues of $A - L^{ss}CA$. The former fact implies that, with no loss of optimality, we may replace the assumption that the encoder has access to the state of the plant at all times by the assumption that the encoder knows only the initial state. The latter implies that if the initial plant state is unknown, then there will be a transient cost that becomes negligible for sufficiently large values of k .

VII. INFINITE HORIZON PROBLEMS WITH TRANSIENT COST

The communication and control strategies presented above are optimal at a specified terminal time, but are only defined over a finite horizon, and may exhibit poor transient performance. It is therefore of interest to consider strategies defined over an infinite horizon, and to evaluate the transient performance of such strategies. We do so in several steps. First we assume that the encoder has access to the state of the plant and transmits a channel input proportional to a *constant* linear combination of the plant states, with the proportionality constant adjusted so that the channel power

constraint is satisfied with equality. We then use the resulting channel output sequence as input to an optimal estimator for the linear combination of states to be transmitted over the channel at the next time step. The covariance of the estimation error will be seen to satisfy a Riccati-like difference equation that depends on the signal to noise ratio (SNR) of the communication channel. Under appropriate hypotheses, the solution to this difference equation converges to that of an associated algebraic equation, and yields a constant and stabilizing estimator gain. At the same time we apply a control input that sets the expected value of the channel input at the next time step equal to zero. Except in special cases, this control input does not set the plant output equal to the conditional output estimation error, and we state a version of Corollary V.6 applicable to the infinite horizon case. Finally we note, as in Section VI, that if the plant is minimum phase and has relative degree one, then an estimator may be used to replace the assumption that the encoder has access to the states of the plant.

Infinite horizon problems with transient penalty on both the state and the control are considered in [7], [8], [30]. The authors of [30] consider a discrete-time LQG problem wherein feedback control is implemented using a vector channel, with one channel per state of the system to be controlled; in the present paper, we consider only a scalar channel. The authors of [7] consider a discrete-time LQG problem with a scalar channel that has noise-free feedback from the channel output to the encoder, and those of [8] consider a continuous-time version of this problem. They show that a separation exists between the tasks of control and communication. As we have already seen, without the noise-free feedback path, there is no such separation.

With an infinite control horizon, it is necessary to insure that the feedback system is stable. Denote the unstable eigenvalues of A by $\{\phi_i, i = 1, \dots, m\}$. The authors of [14] consider the problem of stabilizing the system (1)-(2) over the communication channel (3) using a causal, but possibly nonlinear and time-varying, encoder and decoder/controller. (A preliminary version of [14] is found in [13].) They show that a necessary condition for the feedback system to be *mean square stable* (i.e., $\sup_k \mathcal{E}\{\|X_k\|^2\} < \infty$) is that the channel SNR satisfies the lower bound

$$\mathcal{P}/\sigma_n^2 > -1 + \prod_{i=1}^m |\phi_i|^2. \quad (67)$$

The capacity of a discrete-time Gaussian channel is determined by the SNR of the channel through the formula $\mathcal{C} = (1/2)\log_2(1 + \mathcal{P}/\sigma_n^2)$ [9]. It thus follows from (67) that a necessary condition for stabilization is that the channel capacity satisfy the lower bound $\mathcal{C} > \sum_{i=1}^m \log_2 |\phi_i|$. This formula for channel capacity has been derived in a variety of contexts for Gaussian and noise-free digital channels (e.g. [6], [25], [29]). It is shown in [6] that if the disturbance is not present in (1), then the lower bound (67) can be achieved arbitrarily closely using a unity decoder and a channel input that is a constant linear combination of the states of the plant. If the plant is minimum phase and has relative degree one,

then the same result may be obtained by passing the plant output through a linear time-invariant filter.

If a disturbance is present, then a procedure described in [11], [12] may be used to show that stabilization is possible for any SNR that satisfies the bound (67); however, the mean square norm of the system output may become very large. A number of authors have shown that imposing a performance requirement will require a larger channel capacity or SNR than that required only for stabilization [7], [22], [30]. To illustrate, let us consider the case of a scalar system $x_{k+1} = Ax_k + u_k + d_k$, $y_k = x_k$, where u_k depends on the output of a Gaussian channel, as in (3)-(5). Suppose we want the mean square norm of the state to satisfy the performance bound $\sup_k \mathcal{E}\{X_k^2\} < D$. Then [14, eqn. (21)] implies that

$$D \geq \frac{\sigma_d^2}{1 - |A|^2/2^{2\mathcal{C}}}, \quad (68)$$

and thus the performance specification can be satisfied *only if* $D > \sigma_d^2$. This fact is a consequence of causality: the control at time k can depend only on past values of the disturbance sequence. Since the latter is assumed to be i.i.d., it follows that u_k can do nothing to attenuate the effect of d_k upon the state of the plant. Rearranging (68) yields

$$\mathcal{C} \geq \log_2 |A| - (1/2)\log_2(1 - \sigma_d^2/D), \quad (69)$$

and we see that the capacity necessary for stabilization becomes unbounded as D approaches σ_d^2 . A lower bound on capacity for a disturbance with $\sigma_d^2 = 1$ is derived in [7]; for a scalar plant, it follows from Theorem 4.8 and Remark 4.9 (iv) of [7] that the capacity required for stabilization must satisfy the bound $\mathcal{C} \geq \log_2 |A| - (1/2)\log_2(2\pi e D)$, which is less informative than (69). Finally, it is shown in [14] that the bound (68) is tight in that it is achievable using (linear) communication and control strategies. Generalizations of (68), and thus of (69), to higher order plants are available; however, as discussed in [14], one would not expect the generalized bounds to be tight.

We now proceed to the main results of this section. Assume temporarily that the encoder has access to the state of the plant, choose a row vector H such that $HB \neq 0$, and define channel input and control sequences by

$$s_k = \lambda_k z_k, \quad z_k = Hx_k \quad (70)$$

$$u_k = -F\hat{x}_{k|k}, \quad F \triangleq (HB)^{-1}HA. \quad (71)$$

The state estimate $\hat{x}_{k|k}$ is obtained at the decoder from the recursions

$$\begin{aligned} \hat{x}_{k+1|k} &= A\hat{x}_{k|k} + Bu_k, \\ \hat{x}_{k|k} &= \hat{x}_{k|k-1} + L_k(r_k - \lambda_k H\hat{x}_{k|k-1}), \end{aligned}$$

where the estimator gain is given by

$$L_k = \lambda_k \Sigma_{k|k-1} H^T / (\lambda_k^2 H \Sigma_{k|k-1} H^T + \sigma_n^2), \quad (72)$$

and $\Sigma_{k|k-1}$ satisfies the Riccati difference equation

$$\begin{aligned} \Sigma_{k+1|k} &= A \Sigma_{k|k-1} A^T \\ &\quad - \frac{\lambda_k^2 A \Sigma_{k|k-1} H^T H \Sigma_{k|k-1} A^T}{\lambda_k^2 H \Sigma_{k|k-1} H^T + \sigma_n^2} + \sigma_d^2 E E^T, \end{aligned} \quad (73)$$

with initial condition $\Sigma_{0|-1}$.

Corresponding to Lemma II.1 we have that, with the control (71) applied, $\hat{z}_{k+1|k} = 0$ and $z_{k+1} = \hat{z}_{k+1|k}$, where $\hat{z}_{k+1|k} = \mathcal{E}_{r^k}\{Z_{k+1}\}$ and $\tilde{z}_{k+1|k} = z_k - \hat{z}_{k+1|k}$. It follows that the channel input satisfies $s_k = \lambda_k \tilde{z}_{k|k-1}$, and thus $\mathcal{E}\{S_k^2\} = \lambda_k^2 \mathcal{E}\{\tilde{Z}_{k|k-1}^2\} = \lambda_k^2 H \Sigma_{k|k-1} H^T$.

Suppose that λ_k is adjusted so that $\lambda_k^2 H \Sigma_{k|k-1} H^T = \mathcal{P}$. Then (73) reduces to

$$\Sigma_{k+1|k} = A \Sigma_{k|k-1} A^T - \frac{A \Sigma_{k|k-1} H^T H \Sigma_{k|k-1} A^T}{H \Sigma_{k|k-1} H^T} \frac{\mathcal{P}}{\mathcal{P} + \sigma_n^2} + \sigma_d^2 E E^T. \quad (74)$$

We shall refer to (74) as an ‘‘SNR constrained’’ Riccati difference equation. The corresponding SNR constrained algebraic Riccati equation is given by

$$\Sigma = A \Sigma A^T - \frac{A \Sigma H^T H \Sigma A^T}{H \Sigma H^T} \frac{\mathcal{P}}{\mathcal{P} + \sigma_n^2} + \sigma_d^2 E E^T. \quad (75)$$

Properties of standard Riccati equations are well known [1], including the conditions for existence of a stabilizing solution to the algebraic equation, and for convergence of the solution to the difference equation to that of the algebraic equation. We now develop similar properties for the SNR constrained Riccati equations (74) and (75). Our first result provides conditions under which (75) has a unique solution that yields a stable state estimator.

Proposition VII.1 *Assume that (A, E) is stabilizable, (A, H) is detectable, and that the channel SNR satisfies the lower bound (67). Then the SNR constrained algebraic Riccati equation (75) has a unique positive semidefinite solution $\bar{\Sigma}$ with the property that $H \bar{\Sigma} H^T > 0$. Define $\bar{\lambda}^2 = \mathcal{P}/H \bar{\Sigma} H^T$. Then the eigenvalues of $A - \bar{\lambda} A \bar{L} H$ lie inside the open unit circle, where*

$$\bar{L} = \frac{1}{\bar{\lambda}} \frac{\bar{\Sigma} H^T}{H \bar{\Sigma} H^T} \frac{\mathcal{P}}{\mathcal{P} + \sigma_n^2}. \quad (76)$$

Proof: Define $\lambda^2 \triangleq \mathcal{P}/H \Sigma H^T$, substitute into (75), and multiply both sides of the result by λ^2 , yielding

$$\lambda^2 \Sigma = \lambda^2 A \Sigma A^T - \frac{\lambda^4 A \Sigma H^T H \Sigma A^T}{\lambda^2 H \Sigma H^T + \sigma_n^2} + \lambda^2 \sigma_d^2 E E^T. \quad (77)$$

Define $\Xi \triangleq \lambda^2 \Sigma$, and substitute into (77), resulting in a standard Riccati equation

$$\Xi = A \Xi A^T - \frac{A \Xi H^T H \Xi A^T}{H \Xi H^T + \sigma_n^2} + \lambda^2 \sigma_d^2 E E^T. \quad (78)$$

The solution to (78) is the steady state covariance of the estimation error for the state of the system $\xi_{k+1} = A \xi_k + \lambda E d_k$ based on the noisy measurement $r_k = H \xi_k + n_k$. We now study the dependence of Ξ upon the parameter λ^2 .

Suppose first that $\lambda^2 = 0$ in (78). Then it follows from the proof of Theorem III.1 in [6] that (78) has a solution Ξ_0 that satisfies $H \Xi_0 H^T = \sigma_n^2 (-1 + \prod_{i=1}^m |\phi_i|^2)$. The assumption that (67) is satisfied implies that $H \Xi_0 H^T < \mathcal{P}$. Next suppose that $\lambda^2 > 0$. The assumptions of stabilizability and detectability imply that (75) has a unique positive semidefinite solution

Ξ_λ [1], and that all the eigenvalues of $A - A L_\lambda H$, where $L_\lambda \triangleq \Xi_\lambda H^T / (H \Xi_\lambda H^T + \sigma_n^2)$, lie inside the open unit circle.

The steady state variance of the estimation error for the linear combination of states $H \xi_k$ is given by $H \Xi_\lambda H^T$. An alternate expression may be obtained from [11, eqn. (34)], and is given by

$$H \Xi_\lambda H^T = \sigma_n^2 \left(-1 + \prod_{i=1}^m |\phi_i|^2 e^{2I_\infty(r; \lambda d)} \right),$$

where $I_\infty(r; \lambda d)$, the mutual information rate [22] between the disturbance and the measurement, is equal to

$$I_\infty(r; \lambda d) \triangleq \frac{1}{4\pi} \int_{-\pi}^{\pi} \log_e \left(1 + |G_d(e^{j\omega})|^2 \frac{\lambda^2 \sigma_d^2}{\sigma_n^2} \right) d\omega.$$

It follows immediately that $H \Xi_\lambda H^T$ is a monotonically increasing and unbounded function of λ . Hence there exists a value of λ , call it $\bar{\lambda}$, for which $H \Xi_{\bar{\lambda}} H^T = \mathcal{P}$.

Substitute $\bar{\lambda}$ into (78) and define $\bar{\Sigma} = \Xi_{\bar{\lambda}} / \bar{\lambda}^2$. Then $H \bar{\Sigma} H^T = \mathcal{P} / \bar{\lambda}^2 > 0$, and working backwards yields

$$\bar{\Sigma} = A \bar{\Sigma} A^T - \frac{A \bar{\Sigma} H^T H \bar{\Sigma} A^T}{H \bar{\Sigma} H^T} \frac{\mathcal{P}}{\mathcal{P} + \sigma_n^2} + \sigma_d^2 E E^T.$$

It follows that $\bar{\Sigma}$ obtained as described above is the unique positive semidefinite solution to the SNR constrained Riccati equation (75), and that $A - \bar{\lambda} A \bar{L} H$ has stable eigenvalues. ■

Corollary VII.2 *Assume, in addition to the hypotheses of Proposition VII.1, that (A, E) is reachable. Then $\bar{\Sigma}$, the unique positive semidefinite solution to (75), is in fact positive definite.* ■

Our next step is to show that, under appropriate hypotheses, the solution of the difference equation (74) converges to the positive definite solution of (75). We first present a series of preliminary results. Define

$$RDE(X) \triangleq A X A^T - \frac{A X H^T H X A}{H X H^T} \frac{\mathcal{P}}{\mathcal{P} + \sigma_n^2} + \sigma_d^2 E E^T. \quad (79)$$

Lemma VII.3 *Assume that X_1 and X_2 are symmetric matrices satisfying $X_1 \geq X_2 > 0$. Then $RDE(X_1) \geq RDE(X_2)$.*

Proof: If X is invertible then (79) is equivalent to

$$RDE(X) = A \left(X^{-1} + \frac{\mathcal{P}}{\sigma_n^2} \frac{H^T H}{H X H^T} \right)^{-1} A^T + \sigma_d^2 E E^T. \quad (80)$$

The assumption that $X_1 \geq X_2$ implies that $X_2^{-1} \geq X_1^{-1}$ and $(H^T X_2 H)^{-1} \geq (H^T X_1 H)^{-1}$. The result follows by using these inequalities in (80). ■

Lemma VII.4 *Consider the SNR-constrained Riccati difference equation (74) with initial condition $\Sigma_{0|-1} > 0$. (i) Suppose that $\Sigma_{1|0} \geq \Sigma_{0|-1}$. Then $\Sigma_{k+1|k} \geq \Sigma_{k|k-1}$, $\forall k \geq 0$. (ii) Suppose that $\Sigma_{1|0} \leq \Sigma_{0|-1}$. Then $\Sigma_{k+1|k} \leq \Sigma_{k|k-1}$, $\forall k \geq 0$.*

Proof: (i) It follows from Lemma VII.3 that if $\Sigma_{k+1|k} \geq \Sigma_{k|k-1}$, then $RDE(\Sigma_{k+1|k}) \geq RDE(\Sigma_{k|k-1})$, and thus

$\Sigma_{k+2|k+1} \geq \Sigma_{k+1|k}$. The result follows by mathematical induction and the assumption that $\Sigma_{1|0} \geq \Sigma_{0|-1}$. The proof of (ii) is analogous. ■

Lemma VII.5 Consider the SNR-constrained Riccati difference equation (74) and the associated algebraic Riccati equation (75). Assume that the hypotheses of Corollary VII.2 are satisfied, so that (75) has a unique positive definite solution $\bar{\Sigma} > 0$. (i) Suppose that $\Sigma_{0|-1} \leq \bar{\Sigma}$. Then $\Sigma_{k+1|k} \leq \bar{\Sigma}$, $\forall k \geq 0$.

Proof: (i) Since $\bar{\Sigma} = RDE(\bar{\Sigma})$, it follows from Lemma VII.4 that if $\Sigma_{k|k-1} \leq \bar{\Sigma}$, then $\Sigma_{k+1|k} \leq \bar{\Sigma}$. Hence the result follows by mathematical induction and the assumption that $\Sigma_{0|-1} \leq \bar{\Sigma}$. The proof of (ii) is analogous. ■

We next use these lemmas to prove our main convergence result. We note that the proof is not a straightforward extension to that of the analogous result for standard Riccati equations [1].

Proposition VII.6 Consider the SNR-constrained Riccati difference equation (74) and the associated algebraic Riccati equation (75). Assume that (A, E) is reachable, (A, H) is detectable, and that the channel SNR satisfies the bound (67). Then, for any initial condition $\Sigma_{0|-1}$, the sequence $\Sigma_{k+1|k}$ defined by (74) converges to $\bar{\Sigma}$.

Proof: The assumption that (A, E) is reachable implies that the solution to (74) will be positive definite after at most n time steps. Hence, with no loss of generality, we assume that $\Sigma_{0|-1} > 0$ and establish upper and lower bounds on $\Sigma_{k+1|k}$ that each converge to $\bar{\Sigma}$. To do so, choose $\alpha^- \in (0, 1]$ and $\alpha^+ \in [1, \infty)$ such that $\alpha^- \bar{\Sigma} < \Sigma_{0|-1} < \alpha^+ \bar{\Sigma}$, and define initial conditions $\Sigma_{0|-1}^- \triangleq \alpha^- \Sigma_{0|-1}$ and $\Sigma_{0|-1}^+ \triangleq \alpha^+ \Sigma_{0|-1}$. Denote the corresponding solutions to the Riccati equation (74) by $\Sigma_{k+1|k}^-$ and $\Sigma_{k+1|k}^+$, respectively. By construction, $\Sigma_{0|-1}^- \leq \Sigma_{0|-1} \leq \Sigma_{0|-1}^+$, and thus Lemma VII.3 implies that

$$\Sigma_{k+1|k}^- \leq \Sigma_{k+1|k} \leq \Sigma_{k+1|k}^+, \quad \forall k \geq 0. \quad (81)$$

It remains to show that the bounding sequences in (81) converge to $\bar{\Sigma}$. Note that

$$\begin{aligned} \Sigma_{1|0}^+ &= \alpha^+ \left(A \bar{\Sigma} A^T - \frac{\mathcal{P}}{\mathcal{P} + \sigma_n^2} \frac{A \bar{\Sigma} H H^T \bar{\Sigma} A^T}{H \bar{\Sigma} H^T} \right) + \sigma_d^2 E E^T \\ &= \alpha^+ \bar{\Sigma} + (1 - \alpha^+) \sigma_d^2 E E^T, \end{aligned}$$

and thus $\Sigma_{1|0}^+ \leq \Sigma_{0|-1}^+$ because $\alpha^+ \geq 1$. Similarly, one can show that $\Sigma_{1|0}^- \geq \Sigma_{0|-1}^-$. Therefore, by Lemma VII.5, $\Sigma_{k+1|k}^+$ is a monotonically non-increasing sequence bounded below by $\bar{\Sigma}$, and similarly, $\Sigma_{k+1|k}^-$ is a monotonically non-decreasing sequence bounded above by $\bar{\Sigma}$. Hence both sequences converge, and since the assumptions imply that (75) has a unique positive definite solution $\bar{\Sigma}$, both sequences must converge to $\bar{\Sigma}$. The result then follows from (81). ■

The following result collects a number of facts about the steady state output and output estimation error; part (a) corresponds to Corollary V.6 for the finite horizon case.

Corollary VII.7 Assume that the hypotheses of Proposition VII.6 are satisfied.

(a) In the limit as $k \rightarrow \infty$, the system output becomes stationary and

$$\mathcal{E}\{Y_{k+1}^2\} = C \bar{\Sigma} C^T + C(A - BF) \bar{\Gamma} (A - BF)^T C^T, \quad (82)$$

where F is defined by (71), $\bar{\Gamma}$ satisfies the algebraic Lyapunov equation

$$\bar{\Gamma} = (A - BF) \bar{\Gamma} (A - BF)^T + \frac{\bar{\lambda}^2 \bar{\Sigma} H^T H \bar{\Sigma}}{\mathcal{P} + \sigma_n^2}, \quad (83)$$

and $\bar{\lambda}^2 H \bar{\Sigma} H^T = \mathcal{P}$.

(b) Suppose further that the channel SNR satisfies the lower bound $\mathcal{P}/\sigma_n^2 > \rho^2(A) - 1$, where $\rho(A)$ denotes the spectral radius of A . Then the variance of the steady state output estimation error satisfies the lower bound

$$C \bar{\Sigma} C^T \geq \sigma_d^2 \sum_{\ell=0}^{\infty} (C A^\ell E)^2 \left(\frac{\sigma_n^2}{\mathcal{P} + \sigma_n^2} \right)^\ell. \quad (84)$$

(c) Assume that (A, H) is observable, and that $G_d(z)$ is minimum phase and has relative degree equal to one. Then, in the limit as $\mathcal{P}/\sigma_n^2 \rightarrow \infty$, the output estimation error satisfies $C \bar{\Sigma} C^T = \sigma_d^2 (CE)^2$, independently of the choice of H .

Proof: The identity (a) follows by adapting Corollary V.6 to the infinite horizon case. The lower bound (b) follows by noting that $C \bar{\Sigma} C^T = \lim_{k \rightarrow \infty} \mathcal{E}\{\hat{Y}_{k+1|k}^2\}$, and taking the limit of the right hand side of (57). To prove (c), we note that in the limit as $\mathcal{P}/\sigma_n^2 \rightarrow \infty$, the SNR limited Riccati equation (75) reduces to a standard Riccati equation in the case of perfect measurements. The assumptions of relative degree one and minimum phase imply that the solution to (75) has the form $\bar{\Sigma} = \sigma_d^2 E E^T$ [27], and (c) follows. ■

We note that the lower bound (57) is achievable for any finite value of k using time varying communication and control. The lower bound (84), on the other hand, may not be achievable; indeed, it remains finite for SNRs that are smaller than the minimum (67) required for stabilization. As $\mathcal{P}/\sigma_n^2 \rightarrow \infty$, (84) converges to $\sigma_d^2 (CE)^2$, and is thus achievable by the result of Corollary VII.7 (c).

Example VII.8 Consider the linear system (58) discussed in Example V.7. In that example, we considered the optimal control and communication strategies (46)-(47) for terminal time $N + 1 = 21$. The plots in Figure 8 show that the optimal terminal time strategy has poor transient response, both in terms of the optimal estimation error and the mean square value of the system output. Suppose instead that we apply the strategies (70)-(71), with $H = C$. Then F defined in (71) satisfies $F = (CB)^{-1}CA$, and it follows from the proof of Lemma II.1 that $\mathcal{E}\{Y_{k+1}^2\} = \mathcal{E}\{\hat{Y}_{k+1|k}^2\}$. The plot of $\mathcal{E}\{Y_{k+1}^2\}$ with $H = C$ in Figure 9 shows, as expected, that the terminal cost is greater than that achieved with the optimal strategies (46)-(47); on the other hand, a comparison with Figure 8 shows that the transient response is much improved. Other choices of H are also possible, however, it will no longer be true that

$\mathcal{E}\{Y_{k+1}^2\} = \mathcal{E}\{\tilde{Y}_{k+1|k}^2\}$. For example, plots of $\mathcal{E}\{Y_{k+1}^2\}$ and $\mathcal{E}\{\tilde{Y}_{k+1|k}^2\}$ are also depicted in Figure 9 for the case $H = CA$. In this particular case, the latter choice of H yields values of both $\mathcal{E}\{Y_{k+1}^2\}$ and $\mathcal{E}\{\tilde{Y}_{k+1|k}^2\}$ that are both smaller than those obtained with $H = C$.

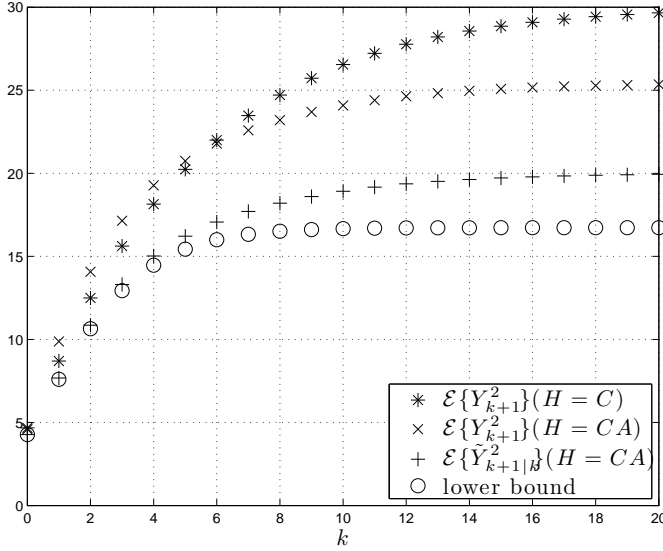


Fig. 9. Plots of $\mathcal{E}\{Y_{k+1}^2\}$ and $\mathcal{E}\{\tilde{Y}_{k+1|k}^2\}$ vs. k for terminal time $N+1 = 21$ using the communication and control strategies (70)-(71), with $H = C$ and $H = CA$. Also plotted is the lower bound (57) on $\mathcal{E}\{Y_{k+1|k}^2\}$.

It is not, in general, true that the performance achieved with $H = CA$ is better than that achieved with $H = C$. To illustrate, in Figure 10 we compare the steady state performance achieved with both strategies as a function of channel SNR. Also shown in Figure 10 is a plot of the limit (84), which is a lower bound on the performance achievable with any communication and control strategies. We note that the scheme $H = CA$ outperforms $H = C$ for low SNRs, including the case $\mathcal{P}/\sigma_n^2 = 2$, which corresponds to Figure 9. For high SNRs, the estimation error for both cases converges to the theoretical minimum, given by $\sigma_d^2(CE)^2$, and as the SNR decreases to the minimum (67) required for stabilization, the response becomes unbounded. Note that the lower bound (84) remains finite for SNRs smaller than that given by (67). This is not a contradiction, because (84) is obtained by taking the limit of the lower bound (57), which is known to be tight only for finite values of k . ■

The last step is to remove the assumption that the encoder has access to the state of the plant. This may be done as in Proposition VI.2. Over an infinite horizon, the effect of a mismatch between the initial conditions of the plant and estimator converges to zero.

VIII. CONCLUSIONS

We have derived optimal communication and control strategies for the problem of minimum variance control with a terminal constraint only, and shown that the optimal strategies are linear and time varying. We have also derived suboptimal strategies that may be implemented over an infinite horizon,

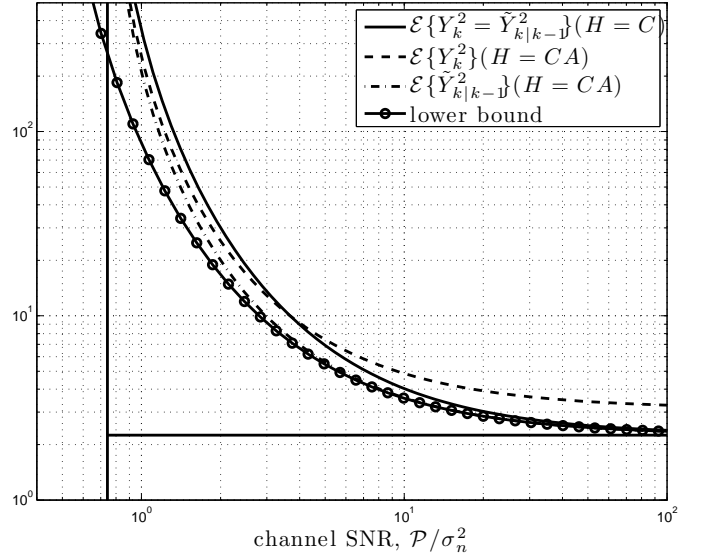


Fig. 10. Plots of steady state $\mathcal{E}\{Y_{k+1}^2\}$ and $\mathcal{E}\{\tilde{Y}_{k+1|k}^2\}$ vs. SNR using the communication and control strategies (70)-(71), with $H = C$ and $H = CA$. Also plotted is the lower bound (84) on $\mathcal{E}\{Y_{k+1|k}^2\}$.

and provided a set of conditions under which these strategies are stabilizing. Our derivations reveal interesting connections between different bodies of previous literature. In particular, the fact that the optimal control input at time k sets the expected value of z_{k+1} equal to zero is well known in the literature on minimum variance control (e.g., [2, Section 6.2], [16, Section 10.3.1]), although the result is usually stated for stationary systems. The design procedure known as “discrete-time loop transfer recovery” [21], [31] uses a controller that sets the system output equal to the conditional estimation error [11, Prop. III.1], and thus also sets the expected value of the output at the next time step equal to zero.

We have noted that, for the purpose of minimizing the terminal cost (6), there is no separation between the tasks of communication and control, but that also there is no conflict. The latter fact is because the sole purpose of control until the final time step is to improve the estimate of the output available at the final time step, and that the last control input sets the output equal to the estimation error. The lack of conflict arises because a performance penalty is imposed only at the final time step. For finite horizon problems with a performance penalty imposed upon the transient response, a conflict between communication and control will arise. Although we do not present an optimal solution to such problems, we note that a solution to the problem (6) yields a lower bound on the optimal value of such a cost, namely

$$\inf_{f_{k,gk}} \sum_{k=0, \dots, N} \mathcal{E}\{Y_{\ell+1}^2\} \geq \sum_{\ell=0}^N J_{\ell+1}^*,$$

where $J_{\ell+1}^*$ denotes the optimal solution to (6) with $N = \ell$.

Finally, in the interest of isolating the performance limitations imposed by the Gaussian communication channel, we have made many simplifying assumptions that may not be satisfied in practice. For example, we have assumed that the

plant is minimum phase and relative degree one, that there is no penalty imposed on the control signal, and that the channel is memoryless and Gaussian. Removing these assumptions will be the subject of future research.

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